Ph.D. thesis proposal:

Seismic propagation in elastic media:

A study of equations of motion by method of characteristics, principal symbols and Fourier-integral operators

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Abstract

My research will focus on the mathematics of seismic wave theory in elastic media. In particular, dealing with a three-dimensional medium, I will study the three coupled Cauchy's equations of motion in anisotropic inhomogeneous media using: (1) the standard method of characteristics, (2) the principal-symbol analysis, (3) Fourier-integral operator (FIO) theory, and (4) numerical analysis. I have already begun and will continue to apply the method of characteristics to the equations of motion. I have also begun to apply principal-symbol theory (related to Fourier-integral operators) to the equations of motion. From that I should also derive the same characteristics. Symbols should put the equations in a framework amenable to application of Fourier-integral operators. I will also try to apply Fourier-integral operator

theory to simple cases so that FIO results can be tested against other known solutions. Then, I will extend that theory to the inhomogeneous anisotropic case and ideally show that it improves on the results of ray theory. In that general case I expect the FIO theory will reduce the complexity of the equations somewhat. Thus less numerical work will be required to complete the solution than for full numerical analysis of the original equations. That will involve some computer implementation, and comparison to real data over known geology. The FIO modelling results for the inhomogeneous anisotropic case can be checked by various methods, for example, inserting the isotropic elasticity matrix into the algorithm and checking that the solution reduces to the isotropic solution. For at least one test case I will also check the anisotropic inhomogeneous FIO solution against a full numerical solution.

1 Introduction

1.1 Rationale

While analytic solutions are possible for simple geological models, for complex models one needs to use approximate methods. Such methods include numerical methods such as finite element or finite difference methods, or theoretical approximate methods such as the seismic ray theory method.

In the research of the Geomechanics Project, the focus is on seismic ray theory. The choice of that method over numerical methods has been largely arbitrary, though full numerical methods may require more computation time and storage than ray methods and may be less able to model propagation of singularities. So far, the attention of the Geomechanics Project has been mostly on traveltimes and ray trajectories or wavefronts but not on the displacement amplitude. My study will centre on modelling that amplitude which is described in the context of ray theory by the transport equation. I plan to further review existing approaches to amplitude modelling in the literature including those in chapters 5–6 of Červený (2001), and in Aki and Richards (2002) and Kennett (2001). Then I will try to devise a new approach, based on Fourier-integral operators (FIOs), to study the equations of motion. I expect to show that FIOs extend the results of the classical transport equation. As a preliminary to solution of the equations of motion I will apply the method of characteristics to the equations of motion and then demonstrate that a principal-symbol formulation yields the same characteristics. That formulation is linked to, and should facilitate development of, a full analytic FIO solution for simple models and/or approximations. But, for a general case, FIO reduction of the equations of motion may be only to a point where some numerical work is still required but much less than if the original equations were numerically solved. In order to check FIO solutions I will have to perform one full finite element or finite difference numerical solution for at least one test model.

1.2 Literature review

The literature review is distributed throughout the text, most notably in subsections 1.5, 1.6, 1.7, 2.3 and subsubsection 3.1.1. The thesis itself will contain a fuller account, but space limits do not permit me to go further here.

1.3 My previous work

So far I have, as outlined in subsection 2.2 below, applied the method of characteristics to the equations of motion to get a long determinantal eikonal equation. In the case of isotropy this equation reduces to the eikonal equations for P and S waves. This result is promising, but I will have to use numerical methods to solve the long form eikonal equation. I have also, as described in subsection 2.3 below, found a principal-symbol matrix determinant eikonal equation that is one-fifth the length of the first one. This equation also reduces to the eikonal equations for P and S waves for the case of isotropy. I plan to review my work for possible errors that, when corrected, would result in the first one and second one being identical.

As well, in the last year-and-a-half I have done some course study and literature review related to those courses and to my thesis research. The four courses are:

- a lecture course on seismic waves and rays (based on Slawinski, 2003), EASC6177;
- a reading/lecture course on differential geometry (based on Arnold, 1989), EASC6172;
- a reading/lecture course on differential geometry (based on Schutz, 1980), EASC6913;
- a reading/lecture course on functional analysis, Sobolev spaces, partial differential equations, pseudo-differential operators and Fourier-integral operators, and method of characteristics, EASC 6912.

1.4 Inhomogeneity

While simple solutions are possible for a homogeneous isotropic model, a general inhomogeneous elastic medium is described by a fourth-order elasticity tensor $c_{ijkl}(\mathbf{x})$ that depends on position $\mathbf{x}$. In the isotropic case, this generally twenty-one independent component tensor can be reduced to one depending on just $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$, as in the test done at the end of subsection 2.2 below.

1.5 Anisotropy

Anisotropy is the variation of wave velocity with direction due to the elasticity tensor being other than the simplest case of isotropic. It is reviewed in Slawinski (2003), Helbig (1994, 2001), MacBeth (2002) and many others. Wardlaw (2002) and Sheriff and Geldart (1995) review related rock properties.

My analysis of equations of motion, and any seismic amplitude modelling based on that analysis, will have to include the cases of both anisotropy and inhomogeneity. However, the mathematics may first be tested on simple cases with other known solutions to ensure that it works in those instances. Or, as in subsection 2.2 below, the complicated case can be solved and then the elasticity matrix set to that for isotropy in order to see if the solution reduces to that for isotropy.

1.6 Asymptotic solutions/ray theory

1.6.1 Geometrical optics

Geometrical optics was well reviewed in the electrical-engineering literature by Kouyoumjian (1965) and Deschamps (1972), among many others. It is based on the equality of the angles of incidence and reflection first noticed by the Greeks, and on Snell's sine law for refraction, developed in the 17th century.

Fermat's principle, which states that the optical length

$$S(C) = \int_C n \, d\sigma$$

must be stationary for the ray curve C joining two points, embodies both of these principles, and forms the basis for optical ray-tracing schemes.

In terms of seismology,

$$S(C) = \int_C \frac{\mathrm{d}s}{v} \, .$$

This equation can be used for basic ray tracing schemes; however, such schemes based on Fermat's principle provide no amplitude information.

1.6.2 Ray theory

The eikonal and transport equations for ray tracing can be derived from Cauchy's equations of motion by using a high-frequency ansatz (trial solution). The rays, which are characteristics of the eikonal equation and bicharacteristics of the wave equation, can be shown to be equivalent to asymptotic solutions of the wave equation. Related asymptotic math is reviewed in Mysak, 1985. This equivalency forms the basis for Asymptotic Ray Theory (ART) and its many variations including Dynamic Ray Tracing; these have been extensively reviewed by Červený (2001) and Burridge (1976) and many others.

However, ART has several failings, including an inability to model wave effects such as caustics, diffractions, and (for zero-order theory) head waves. First-order theory correctly models head waves. Theories such as those of edge waves (Klem-Musatov and Aisenberg, 1985), which is used in geophysics, and the electrical-engineering theory UAT (uniform asymptotic theory of edge diffraction, Lewis and Boersma, 1969) handle diffraction to some extent. Furthermore, there is at times a large sensitivity to small variations in the definition of the model.

The built-in assumption is that the signal is infinite-frequency. Indeed, the ray theory solution is exact only for infinitely high frequency, but it generally agrees reasonably well with finite frequency experimental data, provided that the properties of the continuum do not vary significantly within the wavelength of the signal. The ratio between a significant length scale of change of properties within the continuum and the wavelength of the signal must be much greater than one.

Also, in terms of differential geometry, a ray is a traveltime geodesic, i.e. a path of stationary traveltime and, quite often but not always, minimum traveltime. Some differential geometry reviewed in Arnold (1989) and Schutz (1980) can be used to study such geodesics, since geodesics are a classical part of both the calculus of variations (e.g., Weinstock, 1974) and differential geometry.

1.7 Equations of motion

Here I follow the derivation procedure in Slawinski (2003). For anisotropic inhomogeneous media, we start with Cauchy's equations of motion,

$$\rho(\mathbf{x})\frac{\partial^2 u_i}{\partial t^2} = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}, \qquad i \in \{1,2,3\}. \tag{1}$$

These equations are combined with the stress-strain equations,

$$\sigma_{ij}(\mathbf{x}) = \sum_{k=1}^{3}\sum_{l=1}^{3} c_{ijkl}(\mathbf{x})\varepsilon_{kl}(\mathbf{x}), \qquad i,j \in \{1,2,3\}. \tag{2}$$

Then we proceed to use the definition of the strain tensor,

$$\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right), \tag{3}$$

to obtain equations of motion in inhomogeneous anisotropic media, which are

$$\begin{aligned}
\rho(\mathbf{x})\frac{\partial^2 u_i}{\partial t^2} &= \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left[\frac{1}{2}\sum_{k=1}^{3}\sum_{l=1}^{3} c_{ijkl}(\mathbf{x})\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)\right] \\
&= \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \frac{\partial c_{ijkl}(\mathbf{x})}{\partial x_j}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) \\
&\quad + \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} c_{ijkl}(\mathbf{x})\left(\frac{\partial^2 u_k}{\partial x_j \partial x_l} + \frac{\partial^2 u_l}{\partial x_j \partial x_k}\right). \qquad i \in \{1,2,3\}.
\end{aligned} \tag{4}$$

These equations of motion, due to symmetries in the elasticity tensor in k and l, reduce to

$$\begin{aligned}
\rho(\mathbf{x})\frac{\partial^2 u_i}{\partial t^2} &= \sum_{k=1}^{3}\sum_{l=1}^{3} D_{ikl}\frac{\partial u_k}{\partial x_l} + \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} c_{ijkl}(\mathbf{x})\frac{\partial^2 u_k}{\partial x_j \partial x_l}, \qquad i \in \{1,2,3\}, \\
&\quad \text{where } D_{ikl} = \sum_{j=1}^{3} \frac{\partial c_{ijkl}(\mathbf{x})}{\partial x_j}.
\end{aligned} \tag{5}$$

This is equivalent to the elastic wave equation studied by Stolk and de Hoop (2002), but without body forces included in the equation.

2 Research topics

The aim of this research project is to solve equations (5) for a general elasticity tensor. Several methods may be used for comparison, including the following two: (1) full numerical solution of at least one test example and (2) the method of characteristics, giving analytical derivation of an eikonal equation, followed by numerical solution of that eikonal equation. Both the standard method of characteristics and a principal-symbol variant should yield the same characteristics. However, another more original aspect of this research will be its application of Fourier-integral operator (FIO) methods to solving equations (5). Such a solution ideally will provide information not present in the high-frequency ansatz-derived eikonal- and transport-equation solution. This information may throw light on hitherto unexplained features in real data, and thus provide an aid to interpretation of other real data that shows such features. The solution should provide information on amplitude, traveltime, and ray trajectory. The next step is to determine whether the amplitude information shows any improvement over that obtained by solution of the transport equation, when both are compared to a reliable numerical solution of a test example of an inhomogeneous fully-elastic medium.

Fourier-integral operators (FIOs) have been known for about half a century. Some of their advantages are that

1. unlike the related theory of pseudo-differential operators, which combined with FIOs form the field of micro-local analysis, they can be applied to the analysis and solution of non-elliptic problems such as equations (5);

2. they handle propagation of singularities very well, and indeed for the 3–D wave equation singularities propagate according to the strong Huygens' principle, according to the applications section of Saint Raymond (1991);

3. they are a powerful tool for the investigation of partial differential equations in terms of symbols, distributions and the Fourier transform. While they also to some extent give a high-frequency approximation, they may yield some new information about the solution of the equations of motion that is not present in ray-theory solutions of the eikonal and transport equations.

2.1 Investigating equations of motion

So far our research group has focused largely on traveltime and ray trajectories. In my research I will solve for traveltime, ray trajectories, and displacement amplitudes, but the emphasis will be on obtaining new results related to the amplitudes. According to Slawinski (2003), the transport equation for inhomogeneous elastic media is derived from equation (5); it reads thus:

$$\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\left[\frac{\partial}{\partial x_j}\left(c_{ijkl}(\mathbf{x})A_l\frac{\partial\psi}{\partial x_k}\right)+c_{ijkl}(\mathbf{x})\frac{\partial A_k}{\partial x_l}\frac{\partial\psi}{\partial x_j}\right]=0\,. \tag{6}$$

I will review the recent literature for existing methods of solving the transport equation so that amplitude information from such methods can be compared to my Fourier-integral operator solutions and, in at least one test case, to a full numerical solution of equations (5). Only if such existing methods seem inadequate will I devise new methods to solve the transport equation or to improve on existing methods. Otherwise I will focus on solving equations (5).

A new method will involve Fourier-integral operators, which I plan to apply starting in Spring 2004. This procedure should prove fruitful since Fourier-integral operators are applicable to non-elliptic problems such as equations (5) and gracefully handle propagation of singularities, and are a powerful modern mathematical tool for analyzing PDEs. I shall apply Fourier-integral operators directly to the analysis of equations of motion (5) rather than to the transport equation. I hope to show that this analysis extends the results of past analyses of the transport equation with new amplitude (and perhaps other) information that better matches subtleties in real data over known geology, and in a finite-difference or finite-element numerical solution of at least one test model.

2.2 Standard method of characteristics

Characteristics, also known as exceptional or critical initial manifolds (Courant and Hilbert, 1962), are defined as follows: if the matrix of coefficients of derivatives in a partial differential equation (PDE) is singular on a curve or surface, it is a characteristic curve or surface. Characteristics can be used to reduce the number of variables to be considered. In the special case of a first-order PDE, along the characteristics the PDE is reduced to a system of ODEs. However, such reduction in number of variables may transform a linear PDE into an equation of fewer variables. Such an equation is a nonlinear one, as we shall see later in this document. Also, an initial solution cannot be specified along a characteristic curve or surface, since, given one displacement value on such a surface, all other values are then determined so we cannot arbitrarily specify a second (or further) one.

When the method is applied to the wave equation for arbitrary inhomogeneity and elliptical velocity dependence it involves

1. ascertaining whether the determinant of a three-by-three matrix of coefficients of first derivatives is non-zero so that the system can be solved, and

2. setting the determinant of the six-by-six matrix of coefficients of second derivatives to zero to obtain the eikonal equation.

My next step is to apply the above method to find the characteristics of Cauchy's equations of motion (5) for inhomogeneous anisotropic media. The characteristic surface is one on which the second-order derivatives of u cannot be determined. The first-order derivatives in equations (5) are determined from the initial conditions.

According to Bos (2003), the characteristic surface is one on which the second-order partial derivatives of u cannot be determined. So let us consider a surface S given in functional form by $t = f(\mathbf{x})$. We then require initial conditions on S that are

$$u[\mathbf{x}, f(\mathbf{x})] = \phi(\mathbf{x}) \tag{7}$$

and

$$\frac{\partial \mathbf{u}}{\partial \mathbf{n}}[\mathbf{x}, f(\mathbf{x})] = \psi(x_1, x_2, x_3)\,. \tag{8}$$

Here the partial derivative is the normal derivative with respect to S. Now the surface S can be written as the level set $f(\mathbf{x}) - t = 0$. Subsequently, the normal vector (not necessarily of unit length) is, with

$$\nabla = \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial t}\right]\,,$$

$$\mathbf{n} = [\nabla f(\mathbf{x}), \nabla(-t)] = [\nabla f(\mathbf{x}), -1]\,.$$

The twelve first-order derivatives of u_i are determined from the initial conditions. I have solved for them explicitly, but I omit that mathematics here for brevity. I may include it in my thesis. As in Bos (2003), the solutions for the first-order derivatives of u_i can be abbreviated, so that

$$\frac{\partial u_i}{\partial t}[\mathbf{x}, f(\mathbf{x})] = \alpha_i(\mathbf{x}) \qquad \text{and} \qquad \frac{\partial u_i}{\partial x_j}[\mathbf{x}, f(\mathbf{x})] = F_{ij}(\mathbf{x})\,. \tag{9}$$

(I have stored the α_i and F_{ij} in the Mathematica notebook file `char1.nb` for future reference if needed.)

The above twelve equations (for $i, j = 1, 2, 3$) can then be differentiated with respect to x_1, x_2, x_3 to obtain, first for terms involving α, nine scalar equations:

$$\frac{\partial \alpha}{\partial x_j}(\mathbf{x}) = \frac{\partial^2 \mathbf{u}}{\partial t \partial x_j} + \frac{\partial^2 \mathbf{u}}{\partial t^2}[\mathbf{x}, f(\mathbf{x})] \frac{\partial \mathbf{f}}{\partial x_j}(\mathbf{x}) \,. \tag{10}$$

Secondly, for terms involving F_{ij}, with the notation

$$\mathbf{F}^{[j]} = \frac{\partial \mathbf{u}}{\partial x_j} = [F_{1j}, F_{2j}, F_{3j}] \ ,$$

I derive

$$\frac{\partial \mathbf{F}^{[j]}}{\partial x_k}(\mathbf{x}) = \frac{\partial^2 \mathbf{u}}{\partial x_k \partial x_j} + \frac{\partial^2 \mathbf{u}}{\partial x_j \partial t}[\mathbf{x}, f(\mathbf{x})] \frac{\partial \mathbf{f}}{\partial x_k}(\mathbf{x}) \,. \tag{11}$$

Now, with j, k ranging from one to three, and three scalar equations for each vector equation, that would be 27 equations, but the equality of mixed partial derivatives reduces it to eighteen equations. Those eighteen equations with the previous sets of nine equations and the original equations of motion (three scalar equations) give a system of thirty scalar equations for the thirty second derivatives of u_i.

For this system of equations not to have a solution for the second partial derivatives, the determinant of the thirty-by-thirty matrix of coefficients of the second partial derivatives must be zero. This matrix is given in terms of elasticity-matrix coefficients rather than elasticity-tensor coefficients in appendix A. The matrix has 54 entries that are one of (or twice one of) the 21 independent elasticity coefficients or are a sum of two of them, plus three $-\rho$'s, plus six f_1's, nine f_2's, and twelve f_3's, and 27 ones, so 111 non-zero terms (but only 26 independent components of those terms) which means 789 zeros. Each f_i is $\partial f / \partial x_i$. The components of that thirty-by-thirty matrix depend on components of the gradient of f and on the elements of the elasticity matrix. That thirty-by-thirty matrix multiplied by a vector of second-order derivatives gives the known terms depending on the first-order derivatives. The determinant of the thirty-by-thirty matrix is given on two pages in appendix B. When set to zero it is a nonlinear first-order PDE, an eikonal equation, for f. The numerical solving of this equation is one target of my research.

As a check, I set the elasticity matrix equal to the isotropic elasticity matrix in terms of λ and μ and evaluated the determinant with Mathematica and set it equal to zero to get

$$\left\{(\lambda+2\mu)\left[\left(\frac{\partial f}{\partial x_1}\right)^2+\left(\frac{\partial f}{\partial x_2}\right)^2+\left(\frac{\partial f}{\partial x_3}\right)^2\right]-\rho\right\}\left\{\rho-\mu\left[\left(\frac{\partial f}{\partial x_1}\right)^2+\left(\frac{\partial f}{\partial x_2}\right)^2+\left(\frac{\partial f}{\partial x_3}\right)^2\right]\right\}=0,$$

which leads to

$$\left(\frac{\partial f}{\partial x_1}\right)^2+\left(\frac{\partial f}{\partial x_2}\right)^2+\left(\frac{\partial f}{\partial x_3}\right)^2=\frac{\rho}{\lambda+2\mu}=\frac{1}{v_p^2}, \tag{12}$$

or

$$\left(\frac{\partial f}{\partial x_1}\right)^2+\left(\frac{\partial f}{\partial x_2}\right)^2+\left(\frac{\partial f}{\partial x_3}\right)^2=\frac{\rho}{\mu}=\frac{1}{v_s^2}. \tag{13}$$

These are the eikonal equations for P and S waves in isotropic inhomogeneous media; thus the material in appendices A and B is not demonstrated to be incorrect. However, the determinant in appendix B seems a bit long, so I will triple-check for typos before attempting numerical solution.

2.3 Principal symbols, and their application to characteristics

As a preliminary to defining the symbol of a differential operator (Rauch, 1991; Renardy and Rogers, 1992), let me first explain the concept of multi-indices. A multi-index is a vector

$$\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$$

which has components that are non-negative integers. The following definitions also apply:

$$|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n, \qquad \alpha!=\alpha_1!\alpha_2!\ldots\alpha_n!\,.$$

For vector $\mathbf{x}=(x_1,x_2,\ldots,x_n)$,

$$\mathbf{x}^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}\,.$$

Partial derivatives can be written as

$$D^{\alpha} = \frac{\partial|\alpha|}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}\,.$$

Then, given a linear differential expression of the form

$$L(\mathbf{x}, D)u = \sum_{|\alpha|\leq m} a_{\alpha}(\mathbf{x})(D)^{\alpha}\,,$$

the symbol of that expression is

$$L(\mathbf{x}, i\xi) := \sum_{|\alpha|\leq m} a_{\alpha}(\mathbf{x})(i\xi)^{\alpha}\,.$$

The principal part of that symbol, also called the principal symbol, is

$$L^{P}(\mathbf{x}, i\xi) := \sum_{|\alpha|=m} a_{\alpha}(\mathbf{x})(i\xi)^{\alpha}\,.$$

2.3.1 Application to characteristics of equations of motion

Rauch (1991) says that a smooth hypersurface Σ is characteristic only if the principal symbol $L^P(\mathbf{x}, i\xi)$ is zero for $\xi \in N^*_{\mathbf{x}}(\Sigma)\backslash 0$, where $N^*_{\mathbf{x}}(\Sigma)$ is a 1–D subspace of $T^*_{\mathbf{x}}(\Sigma)$.I have used this principal-symbol method of determining characteristics to check against the method of characteristics used in the previous subsection. But in the case of the system of PDEs (5) the system is formulated in terms of a matrix operator

$$L_{mi}(\mathbf{x}, D)u_i = \sum_{|\alpha|\leq 2} \frac{a_{\alpha}^{mi}(\mathbf{x})\partial^{|\alpha|}u_i}{\partial x_1^{\alpha^1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}\partial x_4^{\alpha_4}}\,. \tag{14}$$

In that equation m and i range from one to three and $x_4 = t$. The procedure of replacing D^{α} by $(i\xi)^{\alpha}$ and considering only the highest order terms, $|\alpha| = 2$, results in a principal-symbol matrix $\mathbf{L}_P(\mathbf{x}, i\xi)$. (I have not included this matrix here; it is available should readers wish to see it.) Setting the determinant, given in appendix C, of this matrix to zero gives a characteristic or eikonal equation. Again, as in the previous section, it is a nonlinear first-order equation in the ξ_i but is one-fifth as long as the one in the previous section. However, when the isotropic elasticity matrix is inserted it reduces to the eikonal equations for P and S waves as before. Due to the differences in length of appendix B compared to appendix C I suspect there is an error in my Mathematica formulation of the first case, which I will attempt to trace, despite it reducing to the correct equations for the isotropic case.

2.3.2 Principal symbols and the Fourier Transform

Moreover, since the coefficients of equations (5) are constant at a given point $\mathbf{x}$, Renardy and Rogers (1992) say that the symbol can be interpreted in terms of the Fourier transform, with the Fourier transform of $u(\mathbf{x})$ being defined as

$$\hat{u}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(\mathbf{x}) e^{-i\xi\cdot\mathbf{x}} \, \mathrm{d}\mathbf{x} \, .$$

Then $L(i\xi)\hat{u}(\xi)$ is the Fourier transform of $L(D)u(\mathbf{x})$.

Since the concept of symbol and the Fourier transform are both important to FIO methods, this section provides an important bridge between the method of characteristics section and full application of FIO methods.

2.4 Fourier-integral operators and related mathematics

I would like to apply the branch of mathematics known as Fourier-integral operators (FIOs) to analyze equations of motion (5). In preparation for this, in 2003–2004 I took a reading course chaired (and with some lectures) by Len Bos and Andrej Bóna on functional analysis (Zimmer, 1990), partial differential equations (McOwen, 2003), pseudo-differential operators (Saint Raymond, 1991; Treves, 1980a; Wakabayashi, 2000), Fourier-integral operators (Treves, 1980b; Hörmander, 1971; Duistermaat and Hörmander, 1972; Wakabayashi, 2000) and the method of characteristics. After my completion of the standard method of characteristics and principal-symbol variant application to equations (5) I will move from the principal-symbol variant method to applying the full FIO method to equations (5).

2.4.1 Pseudo-differential operators

According to Saint Raymond (1991), a pseudo-differential operator is the extension of partial-differential operators as a calculus of polynomial symbols to pseudo-differential operators (ΨDOs) as a calculus of a more general class of symbols which are not necessarily polynomial and which correspond to operators that are not differential. According to the applications section of Saint Raymond (1991), one application is that singularities for the wave equation propagate according to the strong Huygens' principle, at least for the case of three dimensions of space plus one dimension of time.

In Saint Raymond (1991), the theory of ΨDOs is founded on the theory of distributions and the Fourier transform. Also ΨDOs are best applied to elliptic PDEs.

Application of ΨDOs to ocean acoustics is described in Jensen et al. (2000).

2.4.2 Fourier-integral operators

Fourier-integral operators (Treves, 1980b; Hörmander, 1971) are a further integral-operator extension of pseudo-differential operators. They involve a wider class of operators than for pseudo-differential operator theory which result in a calculus almost as smooth as that of pseudo-differential operators but is applicable to genuinely non-elliptic problems (and also to elliptic problems), whereas basic ΨDO theory is applicable only to elliptic problems and extensions to ΨDO theory have been made only to hypoelliptic problems and not genuinely non-elliptic problems.

Both pseudo-differential operators and Fourier-integral operators can make it possible to handle differential operators with variable coefficients in a way similar to that in which one would handle differential operators with constant coefficients using the Fourier transform. However, in the case of equations (5) the coefficients of the partial derivatives, while variable in space due to inhomogeneity, are constant at a given point over time. Thus, the full generality of FIOs is not required, but a simpler variant with constant coefficient can be used. Also equations (5) are a linear system.

For instance, Hörmander (1971) gives the constant coefficient example of the inhomogeneous Laplace equation, an elliptic equation:

$$\nabla^2 u = f \in C_0^\infty(\mathbb{R}^n)\,. \tag{15}$$

For $n > 2$, this equation is solved by

$$u(x) = -(2\pi)^{-n/2} \int e^{ix\cdot\xi} |\xi|^{-2} \hat{f}(\xi)\,\mathrm{d}\xi\,. \tag{16}$$

In order to solve arbitrary elliptic equations with variable coefficients, we have to consider more general operators of the form

$$Af(x) = (2\pi)^{-n/2} \int e^{ix\cdot\xi} a(x,\xi) \hat{f}(\xi)\,\mathrm{d}\xi\,, \tag{17}$$

in which a behaves as a sum of homogeneous functions when $\xi \to \infty$. These are the pseudo-differential operators. Pseudo-differential operators are extended to general Fourier-integral operators by

$$Af(x) = \int\!\!\int e^{i\phi(x,y,\xi)} a(x,y,\xi) f(y)\,\mathrm{d}y\,\mathrm{d}\xi\,, \tag{18}$$

where $\phi(x,y,\xi) = S(x,\xi) - y\cdot\xi$, and a is independent of y.

In a related formulation, Candès and Demanet (2003) state that an operator T is said to be a Fourier-integral operator if it is of the form

$$Tf(x) = \int e^{i\Phi(x,\xi)} a(x,\xi) \hat{f}(\xi)\,\mathrm{d}\xi\,, \tag{19}$$

where Φ is a phase function and a is an amplitude, and where it is assumed that a is a symbol of order m and Φ is homogeneous in degree 1 in ξ. This is similar to the above general formulation (18) in that Φ is not necessarily a simple inner product, but in this case there is integration just over ξ and not over both ξ and y. These operators are a powerful mathematical tool for analyzing and solving systems of hyperbolic partial-differential equations such as equations (5).

While completing the application of the standard method of characteristics and principal-symbol variant, which will involve numerically solving the respective eikonal equations, I plan to read more on FIO theory and applications. The principal-symbol method for finding the characteristic surface required finding the principal symbol matrix of equations (5). This will be useful in my applying FIOs to equations (5) since FIO theory draws on symbols and the Fourier Transform. This FIO analysis will draw on my study of Sobolev spaces, symbols, functional analysis, partial differential equations and pseudo-differential operators in 2003–2004.

One advantage of FIOs is that they allow an elegant description of the propagation of wavefronts. Equations of motion (5) are a complicated though linear system of three partial-differential equations with constant (in time but not space) coefficients. FIO theory is a powerful modern mathematical theory for the analysis of such equations. However, just applying a new mathematical technique for the sake of testing said mathematical technique, while a worthwhile exercise, is not my complete objective. My aim is threefold: to test the new mathematics by application, to reveal subtleties of modelling not apparent in ray theory solution of the eikonal and transport equations, and to show some advantages over full numerical solution.

2.5 Practical implementation

2.5.1 Computer implementation

Already in applying the standard method of characteristics I have had to use Mathematica to reduce large determinants. The characteristic equation for the equations of motion, while it reduces to P and S wave eikonal equations for the isotropic case, is in the general case a two-page nonlinear first-order equation. I must solve this equation numerically, even if an error is found that when corrected reduces it to the equation one-fifth as long resulting from the principal-symbol variant method discussed earlier.

In the later application of the FIO method to the equations of motion, as in the method of characteristics the process may reach a simpler stage beyond which analytic solutions are not possible and numerical methods must be applied. Except for simple test examples and approximations, the finding of a fully analytic solution that can then be coded onto a computer may be unlikely. The two-stage process of some analytic reduction and some application of numerical methods should result in a much more tractable numerical problem than direct numerical solution of the equations of motion, however. For at least one test example, ideally for known geology above which real data has been collected and is available to me, I will also apply a full numerical method, such as finite element or finite difference, directly to the equations of motion. In this case care is necessary to avoid possible problems with such numerical methods. On the other hand, it might also be possible to demonstrate that a full numerical method has problems in some cases where the FIO method does not. One objective, therefore, is to learn relative advantages and disadvantages of different solution methods by testing.

2.5.2 Comparison to real data

During later stages there will be some comparison of any modelling results (including the three components of displacement) to real three-component seismic data. This could be either purely surface or VSP, over known geology, with some drill-core constraints and ideally known elasticity matrices of the rock layers. Such data I hope would be provided by a supporting company of The Geomechanics Project. The analysis of the equations (5) by

FIO methods in itself, without necessarily finding a solution, may form a portion of my thesis. However, I do plan also to find solutions. As noted above, such findings may be achieved through some analytic reduction followed by completion with numerics. I would compare such solutions to a least one real-data test case and at least one full numerical-solution test case. It should prove true that subtleties of physical phenomena will be better accounted for by the more complicated mathematics applied to the equations of motion than by simpler methods such as ray-theory solutions of the eikonal and transport equations.

3 Originality and timeline

3.1 Originality

The Geomechanics Project has so far focused on ray trajectories and traveltimes within seismic ray theory rather than displacement amplitudes in the context of the transport equation or otherwise. While three-component displacement modelling is only one aspect of my research, it should complement other studies within the Geomechanics Project and extend recent results in the literature on amplitude modelling in fully elastic inhomogeneous media. The proposed application of FIOs to analyzing the equations of motion should extend the results of previous analyses of the transport equation. As well, it is a new approach that may reveal subtleties of modelling. With comparison to standard method of characteristics and principal symbol variant and to full numerical methods for at least one test case and real data over known geology for at least one test case it would form a substantial body of work sufficient for a Ph.D. This research ought to result in some primary publications and conference presentations during the degree years and immediately following degree completion.

3.1.1 Related work of other people

Few researchers have applied Fourier-integral operator theory or similar branches of mathematics to seismic problems. De Hoop and Hörmann (2001, 2003) and Stolk and de Hoop (2002) have done some work applying microlocal analysis, a field of mathematics which includes Fourier-integral operators and pseudo-differential operators, to seismic problems. The elastic-wave equation they use is equivalent to the equations of motion (5) but with body forces. They use pseudo-differential operators and Fourier-integral operators to do forward modelling with the Kirchhoff and Born approximations separately and together, and also do inverse scattering. Their mathematical development should smooth my way somewhat, but there is some room for further original developments beyond their work, perhaps again in analytic development to a point, followed by numerical completion, without such approximations. However their work may mean that on its own my application of FIO methods to equations (5) might not be quite enough for a Ph.D. but when combined with the method of characteristics, principal-symbol analysis, and comparison to numerical results and real data for one test case it should be enough. If not I could also add a consideration of nonlinearity and possibly anelasticity.

3.2 Timeline

- By the end of April, 2004, I will verify that the principal-symbol variant method results in the same characteristic equation as the standard method of characteristics.

- During May, 2004 I will code a numerical solution of the characteristic (eikonal) equation. I will also learn more about FIO mathematics and extend my literature review on FIO applications. I may also submit an abstract to the IWSA conference by May 31.

- From June to August, 2004 I will build on the principal-symbol work by applying the FIO method (which includes use of symbols) to the equations of motion. In late July 2004, I will meet with de Hoop (and Stolk if he is here) and other scientists visiting St. John's for the 11th IWSA conference.

- In 2004–2005 I will continue the application of FIO methods to the equations of motion and code up fully analytic solutions of simple cases and also numerical solution of a stage I anticipate FIO methods may reduce the equations of motion to, beyond which an analytic solution will not be possible except for such simple cases/approximations. This approach still should yield some advantages over full numerical solution of the equations of motion. In late stages I will also code up (or build on existing code of others) a full numerical solution, using finite element or finite difference method, to at least one test model for the equations of motion. I will also compare results, especially three-component displacement, to real data obtained above known geology. Ideally that would be the model used for the full numerical solution as well. Also, I hope to apply existing code, written by others, for solving the eikonal and transport equations to the same model. There would thus be comparisons for the same model of: (1) real data, (2) full numerical solution, (3) method of characteristics followed by numerical solution of the eikonal equation, (4) existing methods for solving the transport and eikonal equation, and (5) FIO methods with at least some analytic reduction of the equations of motion followed by, in complicated cases, probably some numerical work to get the final solution.

- The earliest date of completion is early September, 2005. This date is unlikely but possible if all goes well and if I put off submitting papers for publication until after thesis submission. If I submit a couple of primary publications during the course of the thesis, there are difficulties in the analytic math or numerical coding or if I follow a lead for a month or so and then it turns out not to pan out and I have to start a new course, then early September, 2006, four years after I started, is more likely. However, I would prefer to graduate by May 2006 convocation if I can.

4 Logistical and financial support

4.1 Financial support

I will be funded at least until September, 2005, mainly by an annual Departmental graduate fellowship, consisting of a combination of money from The Geomechanics Project and the Department, with a smaller amount of money coming from a teaching assistantship. Should I get a scholarship equalling or exceeding that combined funding, the scholarship will be my main source and the other funding will be cut off for that year.

4.2 Logistical support

Most of my research will involve applied mathematics and some computer programming, I already have access to most of the tools I need, including books, journals, computer, Mathematica, Fortran compiler, plotting software, and LaTeX and related tools. If later in the course of the Ph.D. thesis it is deemed desirable to check my theoretical results against real data over known geology, perhaps one of The Geomechanics Project's supporting companies would provide it free of charge in exchange for first access to my results of testing such real data against my modelling methods.

My supervisory committee will be here much of the time during the research period and when a member is away interaction can continue via e-mail. I expect to confer with visitors to some degree as well, for instance during the 11th IWSA conference late this coming July. In addition, I can communicate by e-mail and by math newsgroups with remote mathematicians knowledgeable in the area of FIOs, as I have already done to a minor degree. Similar correspondence is possible with remote experts in seismic ray theory and/or anisotropy.

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A The thirty-by-thirty matrix of coefficients

C_{11}	$2C_{16}$	$2C_{15}$	C_{66}	$2C_{56}$	C_{55}	C_{16}	$C_{12}+C_{66}$	$C_{14}+C_{56}$	C_{26}	$C_{25}+C_{46}$	C_{45}	C_{15}	$C_{41}+C_{56}$	$C_{13}+C_{55}$	C_{46}	$C_{36}+C_{45}$	C_{35}	$-\rho$	0	0	0	0	0	0	0	0	0	0	0
C_{16}	$C_{21}+C_{66}$	$C_{14}+C_{56}$	C_{26}	$C_{25}+C_{46}$	C_{45}	C_{66}	$2C_{26}$	$2C_{46}$	C_{22}	$2C_{24}$	C_{44}	C_{56}	$C_{25}+C_{46}$	$C_{36}+C_{45}$	C_{24}	$C_{23}+C_{44}$	C_{34}	0	$-\rho$	0	0	0	0	0	0	0	0	0	0
C_{15}	$C_{14}+C_{56}$	$C_{13}+C_{55}$	C_{46}	$C_{36}+C_{45}$	C_{35}	C_{56}	$C_{25}+C_{46}$	$C_{36}+C_{45}$	C_{24}	$C_{23}+C_{44}$	C_{34}	C_{55}	$2C_{45}$	$2C_{35}$	C_{44}	$2C_{34}$	C_{33}	0	0	$-\rho$	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_1	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_2	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_3	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_1	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_2	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_3	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_1	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_2	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_3	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_1	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_2	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_3	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_2	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_3	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	f_2	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	f_3
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_2	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_3	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_3	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_2	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_3	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	f_3	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	f_2	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	f_3	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	f_3	0

B Determinant of the thirty-by-thirty matrix

$$-(f_2(f_3^7((-C_{46}f_2-(C_{36}+C_{45})f_3)(-C_{35}f_3^2+f_2(-C_{46}f_2-(C_{36}+C_{45})f_3)+f_1(-C_{15}f_1-(C_{14}+C_{56})f_2$$
$$-(C_{13}+C_{55})f_3))+(C_{44}f_2+2C_{34}f_3)(\rho-C_{55}f_3^2+f_1(-C_{11}f_1-2C_{16}f_2-2C_{15}f_3)+f_2(-C_{66}f_2$$
$$-2C_{56}f_3)))(-f_3^2(\rho f_3^2-C_{44}f_3^4)(-C_{35}f_3^2+f_2(-C_{46}f_2+(-C_{36}-C_{45})f_3)+f_1(-C_{15}f_1+(-C_{14}-C_{56})f_2$$
$$+(-C_{13}-C_{55})f_3))-C_{34}f_3^6(-C_{45}f_3^2+f_2(-C_{26}f_2+(-C_{25}-C_{46})f_3)+f_1(-C_{16}f_1+(-C_{12}-C_{66})f_2$$
$$+(-C_{14}-C_{56})f_3))-f_2f_3^4((-C_{22}f_2-2C_{24}f_3)(-C_{35}f_3^2+f_2(-C_{46}f_2+(-C_{36}-C_{45})f_3)+f_1(-C_{15}f_1$$
$$+(-C_{14}-C_{56})f_2+(-C_{13}-C_{55})f_3))-(-C_{24}f_2+(-C_{23}-C_{44})f_3)(-C_{45}f_3^2+f_2(-C_{26}f_2+(-C_{25}-C_{46})f_3)$$
$$+f_1(-C_{16}f_1+(-C_{12}-C_{66})f_2+(-C_{14}-C_{56})f_3)))-f_1f_3^4((-C_{66}f_1-2C_{26}f_2-2C_{46}f_3)(-C_{35}f_3^2$$
$$+f_2(-C_{46}f_2+(-C_{36}-C_{45})f_3)+f_1(-C_{15}f_1+(-C_{14}-C_{56})f_2+(-C_{13}-C_{55})f_3))-(-C_{56}f_1+(-C_{25}-C_{46})f_2$$
$$+(-C_{36}-C_{45})f_3)(-C_{45}f_3^2+f_2(-C_{26}f_2+(-C_{25}-C_{46})f_3)+f_1(-C_{16}f_1+(-C_{12}-C_{66})f_2+(-C_{14}-C_{56})f_3))))$$
$$-f_3^11((-C_{24}f_2+(-C_{23}-C_{44})f_3)(-C_{35}f_3^2+f_2(-C_{46}f_2+(-C_{36}-C_{45})f_3)+f_1(-C_{15}f_1+(-C_{14}-C_{56})f_2$$
$$+(-C_{13}-C_{55})f_3))-(-C_{44}f_2-2C_{34}f_3)(-C_{45}f_3^2+f_2(-C_{26}f_2+(-C_{25}-C_{46})f_3)+f_1(-C_{16}f_1+(-C_{12}-C_{66})f_2$$
$$+(-C_{14}-C_{56})f_3)))(-f_2((C_{24}f_2+(C_{23}+C_{44})f_3)(\rho-C_{11}f_1^2-2C_{16}f_1f_2-C_{66}f_2^2-2C_{15}f_1f_3-2C_{56}f_2f_3$$
$$-C_{55}f_3^2)+(-C_{26}f_2+(-C_{25}-C_{46})f_3)(-C_{35}f_3^2+f_2(-C_{46}f_2+(-C_{36}-C_{45})f_3)+f_1(-C_{15}f_1+(-C_{14}-C_{56})f_2$$
$$+(-C_{13}-C_{55})f_3)))-f_3^2(-C_{45}(-C_{35}f_3^2+f_2(-C_{46}f_2+(-C_{36}-C_{45})f_3)+f_1(-C_{15}f_1+(-C_{14}-C_{56})f_2+(-C_{13}$$
$$-C_{55})f_3))+C_{34}(\rho-C_{55}f_3^2+f_1(-C_{11}f_1-2C_{16}f_2-2C_{15}f_3)+f_2(-C_{66}f_2-2C_{56}f_3)))-f_1((-C_{16}f_1$$
$$+(-C_{12}-C_{66})f_2+(-C_{14}-C_{56})f_3)(-C_{35}f_3^2+f_2(-C_{46}f_2+(-C_{36}-C_{45})f_3)+f_1(-C_{15}f_1+(-C_{14}-C_{56})f_2$$
$$+(-C_{13}-C_{55})f_3))-(-C_{56}f_1+(-C_{25}-C_{46})f_2+(-C_{36}-C_{45})f_3)(\rho-C_{55}f_3^2+f_1(-C_{11}f_1-2C_{16}f_2-2C_{15}f_3)$$
$$+f_2(-C_{66}f_2-2C_{56}f_3)))))+f_1(f_3^7((-C_{15}f_1+(-C_{14}-C_{56})f_2+(-C_{13}-C_{55})f_3)(-C_{35}f_3^2+f_2(-C_{46}f_2+(-C_{36}$$
$$-C_{45})f_3)+f_1(-C_{15}f_1+(-C_{14}-C_{56})f_2+(-C_{13}-C_{55})f_3))-(-C_{55}f_1-2C_{45}f_2-2C_{35}f_3)(\rho-C_{55}f_3^2$$
$$+f_1(-C_{11}f_1-2C_{16}f_2-2C_{15}f_3)+f_2(-C_{66}f_2-2C_{56}f_3)))(-f_3^2(\rho f_3^2-C_{44}f_3^4)(-C_{35}f_3^2+f_2(-C_{46}f_2$$
$$+(-C_{36}-C_{45})f_3)+f_1(-C_{15}f_1+(-C_{14}-C_{56})f_2+(-C_{13}-C_{55})f_3))-C_{34}f_3^6(-C_{45}f_3^2+f_2(-C_{26}f_2+(-C_{25}$$
$$-C_{46})f_3)+f_1(-C_{16}f_1+(-C_{12}-C_{66})f_2+(-C_{14}-C_{56})f_3))-f_2f_3^4((-C_{22}f_2-2C_{24}f_3)(-C_{35}f_3^2$$
$$+f_2(-C_{46}f_2+(-C_{36}-C_{45})f_3)+f_1(-C_{15}f_1+(-C_{14}-C_{56})f_2+(-C_{13}-C_{55})f_3))-(-C_{24}f_2+$$
$$(-C_{23}-C_{44})f_3)(-C_{45}f_3^2+f_2(-C_{26}f_2+(-C_{25}-C_{46})f_3)+f_1(-C_{16}f_1+(-C_{12}-C_{66})f_2+(-C_{14}-C_{56})f_3)))$$
$$-f_1f_3^4((-C_{66}f_1-2C_{26}f_2-2C_{46}f_3)(-C_{35}f_3^2+f_2(-C_{46}f_2+(-C_{36}-C_{45})f_3)+f_1(-C_{15}f_1+(-C_{14}-C_{56})f_2$$
$$+(-C_{13}-C_{55})f_3))-(-C_{56}f_1+(-C_{25}-C_{46})f_2+(-C_{36}-C_{45})f_3)(-C_{45}f_3^2+f_2(-C_{26}f_2+(-C_{25}-C_{46})f_3)$$
$$+f_1(-C_{16}f_1+(-C_{12}-C_{66})f_2+(-C_{14}-C_{56})f_3))))-f_3^11((-C_{56}f_1+(-C_{25}-C_{46})f_2$$

$+(-C_{36}-C_{45})f_3)(-C_{35}f_3^2+f_2(-C_{46}f_2+(-C_{36}-C_{45})f_3)+f_1(-C_{15}f_1+(-C_{14}-C_{56})f_2+(-C_{13}-C_{55})f_3))$
$-(-C_{55}f_1-2C_{45}f_2-2C_{35}f_3)(-C_{45}f_3^2+f_2(-C_{26}f_2+(-C_{25}-C_{46})f_3)+f_1(-C_{16}f_1+(-C_{12}-C_{66})f_2$
$+(-C_{14}-C_{56})f_3)))(-f_2((C_{24}f_2+(C_{23}+C_{44})f_3)(\rho-C_{11}f_1^2-2C_{16}f_1f_2-C_{66}f_2^2-2C_{15}f_1f_3$
$-2C_{56}f_2f_3-C_{55}f_3^2)+(-C_{26}f_2+(-C_{25}-C_{46})f_3)(-C_{35}f_3^2+f_2(-C_{46}f_2+(-C_{36}-C_{45})f_3)$
$+f_1(-C_{15}f_1+(-C_{14}-C_{56})f_2+(-C_{13}-C_{55})f_3)))-f_3^2(-C_{45}(-C_{35}f_3^2+f_2(-C_{46}f_2+(-C_{36}-C_{45})f_3)$
$+f_1(-C_{15}f_1+(-C_{14}-C_{56})f_2+(-C_{13}-C_{55})f_3))+C_{34}(\rho-C_{55}f_3^2+f_1(-C_{11}f_1-2C_{16}f_2-2C_{15}f_3)$
$+f_2(-C_{66}f_2-2C_{56}f_3)))-f_1((-C_{16}f_1+(-C_{12}-C_{66})f_2+(-C_{14}-C_{56})f_3)(-C_{35}f_3^2+f_2(-C_{46}f_2$
$+(-C_{36}-C_{45})f_3)+f_1(-C_{15}f_1+(-C_{14}-C_{56})f_2+(-C_{13}-C_{55})f_3))-(-C_{56}f_1+(-C_{25}-C_{46})f_2$
$+(-C_{36}-C_{45})f_3)(\rho-C_{55}f_3^2+f_1(-C_{11}f_1-2C_{16}f_2-2C_{15}f_3)+f_2(-C_{66}f_2-2C_{56}f_3)))))$
$+f_3(f_3^2(-C_{35}f_3^6(-C_{35}f_3^2+f_2(-C_{46}f_2+(-C_{36}-C_{45})f_3)+f_1(-C_{15}f_1+(-C_{14}-C_{56})f_2$
$+(-C_{13}-C_{55})f_3))-f_3^4(\rho-C_{33}f_3^2)(\rho-C_{55}f_3^2+f_1(-C_{11}f_1-2C_{16}f_2-2C_{15}f_3)$
$+f_2(-C_{66}f_2-2C_{56}f_3)))(-f_3^2(\rho f_3^2-C_{44}f_3^4)(-C_{35}f_3^2+f_2(-C_{46}f_2+(-C_{36}-C_{45})f_3)+f_1(-C_{15}f_1$
$+(-C_{14}-C_{56})f_2+(-C_{13}-C_{55})f_3))-C_{34}f_3^6(-C_{45}f_3^2+f_2(-C_{26}f_2+(-C_{25}-C_{46})f_3)+f_1(-C_{16}f_1+(-C_{12}$
$-C_{66})f_2+(-C_{14}-C_{56})f_3))-f_2f_3^4((-C_{22}f_2-2C_{24}f_3)(-C_{35}f_3^2+f_2(-C_{46}f_2+(-C_{36}-C_{45})f_3)$
$+f_1(-C_{15}f_1+(-C_{14}-C_{56})f_2+(-C_{13}-C_{55})f_3))-(-C_{24}f_2+(-C_{23}-C_{44})f_3)(-C_{45}f_3^2+f_2(-C_{26}f_2$
$+(-C_{25}-C_{46})f_3)+f_1(-C_{16}f_1+(-C_{12}-C_{66})f_2+(-C_{14}-C_{56})f_3)))-f_1f_3^4((-C_{66}f_1-2C_{26}f_2$
$-2C_{46}f_3)(-C_{35}f_3^2+f_2(-C_{46}f_2+(-C_{36}-C_{45})f_3)+f_1(-C_{15}f_1+(-C_{14}-C_{56})f_2+(-C_{13}-C_{55})f_3))$
$-(-C_{56}f_1+(-C_{25}-C_{46})f_2+(-C_{36}-C_{45})f_3)(-C_{45}f_3^2+f_2(-C_{26}f_2+(-C_{25}-C_{46})f_3)+f_1(-C_{16}f_1$
$+(-C_{12}-C_{66})f_2+(-C_{14}-C_{56})f_3))))-f_3^6(-C_{34}f_3^6(-C_{35}f_3^2+f_2(-C_{46}f_2+(-C_{36}-C_{45})f_3)$
$+f_1(-C_{15}f_1+(-C_{14}-C_{56})f_2+(-C_{13}-C_{55})f_3))-f_3^4(\rho-C_{33}f_3^2)(-C_{45}f_3^2+f_2(-C_{26}f_2$
$+(-C_{25}-C_{46})f_3)+f_1(-C_{16}f_1+(-C_{12}-C_{66})f_2+(-C_{14}-C_{56})f_3)))(-f_2((C_{24}f_2+(C_{23}$
$+C_{44})f_3)(\rho-C_{11}f_1^2-2C_{16}f_1f_2-C_{66}f_2^2-2C_{15}f_1f_3-2C_{56}f_2f_3-C_{55}f_3^2)+(-C_{26}f_2$
$+(-C_{25}-C_{46})f_3)(-C_{35}f_3^2+f_2(-C_{46}f_2+(-C_{36}-C_{45})f_3)+f_1(-C_{15}f_1+(-C_{14}-C_{56})f_2$
$+(-C_{13}-C_{55})f_3)))-f_3^2(-C_{45}(-C_{35}f_3^2+f_2(-C_{46}f_2+(-C_{36}-C_{45})f_3)+f_1(-C_{15}f_1+(-C_{14}-C_{56})f_2$
$+(-C_{13}-C_{55})f_3))+C_{34}(\rho-C_{55}f_3^2+f_1(-C_{11}f_1-2C_{16}f_2-2C_{15}f_3)+f_2(-C_{66}f_2-2C_{56}f_3)))$
$-f_1((-C_{16}f_1+(-C_{12}-C_{66})f_2+(-C_{14}-C_{56})f_3)(-C_{35}f_3^2+f_2(-C_{46}f_2+(-C_{36}-C_{45})f_3)+f_1(-C_{15}f_1$
$+(-C_{14}-C_{56})f_2+(-C_{13}-C_{55})f_3))-(-C_{56}f_1+(-C_{25}-C_{46})f_2+(-C_{36}-C_{45})f_3)(\rho-C_{55}f_3^2$
$+f_1(-C_{11}f_1-2C_{16}f_2-2C_{15}f_3)+f_2(-C_{66}f_2-2C_{56}f_3))))))/(f_3^11(-C_{35}f_3^2$
$+f_2(-C_{46}f_2+(-C_{36}-C_{45})f_3)+f_1(-C_{15}f_1+(-C_{14}-C_{56})f_2+(-C_{13}-C_{55})f_3)))$

C Principal symbol characteristic results

$$-2(\xi_1^2C_{15}+\xi_3^2C_{53}+\xi_3(\xi_1(C_{13}+C_{55})+\xi_2(C_{54}+C_{63}))+\xi_2(\xi_2C_{64}+\xi_1(C_{14}+C_{65})))(-4(\xi_1^2C_{15}+\xi_3^2C_{53}$$
$$+\xi_3(\xi_1(C_{13}+C_{55})+\xi_2(C_{54}+C_{63}))+\xi_2(\xi_2C_{64}+\xi_1(C_{14}+C_{65})))(-(\rho\eta^2)+\xi_2^2C_{22}+2\xi_2\xi_3C_{24}+\xi_3^2C_{44}$$
$$+2\xi_1\xi_2C_{62}+2\xi_1\xi_3C_{64}+\xi_1^2C_{66})+4(\xi_3^2C_{43}+\xi_3(\xi_2(C_{23}+C_{44})+\xi_1(C_{54}+C_{63}))+\xi_2(\xi_2C_{24}+\xi_1(C_{52}+C_{64}))$$
$$+\xi_1^2C_{65})(\xi_1^2C_{16}+\xi_3^2C_{54}+\xi_3(\xi_2(C_{52}+C_{64})+\xi_1(C_{14}+C_{65}))+\xi_2(\xi_2C_{62}+\xi_1(C_{12}+C_{66}))))+2(\xi_3^2C_{43}$$
$$+\xi_3(\xi_2(C_{23}+C_{44})+\xi_1(C_{54}+C_{63}))+\xi_2(\xi_2C_{24}+\xi_1(C_{52}+C_{64}))+\xi_1^2C_{65})(4(\xi_3^2C_{43}+\xi_3(\xi_2(C_{23}+C_{44})$$
$$+\xi_1(C_{54}+C_{63}))+\xi_2(\xi_2C_{24}+\xi_1(C_{52}+C_{64}))+\xi_1^2C_{65})(-(\rho\eta^2)+\xi_1^2C_{11}+2\xi_1\xi_3C_{15}+2\xi_1\xi_2C_{16}+\xi_3^2C_{55}$$
$$+2\xi_2\xi_3C_{65}+\xi_2^2C_{66})-4(\xi_1^2C_{15}+\xi_3^2C_{53}+\xi_3(\xi_1(C_{13}+C_{55})+\xi_2(C_{54}+C_{63}))+\xi_2(\xi_2C_{64}+\xi_1(C_{14}+C_{65})))$$
$$(\xi_1^2C_{16}+\xi_3^2C_{54}+\xi_3(\xi_2(C_{52}+C_{64})+\xi_1(C_{14}+C_{65}))+\xi_2(\xi_2C_{62}+\xi_1(C_{12}+C_{66}))))-2(-(\rho\eta^2)+\xi_3^2C_{33}+2\xi_2\xi_3C_{43}$$
$$+\xi_2^2C_{44}+2\xi_1\xi_3C_{53}+2\xi_1\xi_2C_{54}+\xi_1^2C_{55})(4(-(\rho\eta^2)+\xi_2^2C_{22}+2\xi_2\xi_3C_{24}+\xi_3^2C_{44}+2\xi_1\xi_2C_{62}+2\xi_1\xi_3C_{64}$$
$$+\xi_1^2C_{66})(-(\rho\eta^2)+\xi_1^2C_{11}+2\xi_1\xi_3C_{15}+2\xi_1\xi_2C_{16}+\xi_3^2C_{55}+2\xi_2\xi_3C_{65}+\xi_2^2C_{66})-4(\xi_1^2C_{16}+\xi_3^2C_{54}$$
$$+\xi_3(\xi_2(C_{52}+C_{64})+\xi_1(C_{14}+C_{65}))+\xi_2(\xi_2C_{62}+\xi_1(C_{12}+C_{66})))^2)$$